Technical Report 203

TEMPERATURE EFFECTS ON THE HIGH-FREQUENCY STABILITY OF A CYLINDRICAL, RELATIVISTIC ELECTRON BEAM PENETRATING A PLASMA: I

By: G. Dorman

Prepared for:

OFFICE OF NAVAL RESEARCH
WASHINGTON 25, D.C.

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ABSTRACT

This report describes the beginning of a study of the propagation of small-amplitude disturbances in a system consisting of a relativistic electron beam of finite radius penetrating a plasma. The analysis is restricted to those modes satisfying the high-frequency criterion, $|\omega - k V_0|^2 \gg \omega_\beta^2$, where V_0 is the drift velocity and ω_β the betatron frequency of the beam. The main concern of this work will be to ascertain the general effects of beam and plasma temperature upon the stability of these high-frequency modes.

The collisionless Boltzmann equation is employed to calculate the temperature-dependent current response of both the beam and the plasma. The system is assumed to be neutral with uniform densities both inside and outside the beam and, consistent with the high-frequency criterion, the pinch field is neglected. In this report only the $\ell = 0$ normal mode is considered.

Dispersion relations are derived for the cases of an infinite plasma, of a plasma bounded by a cylindrical conducting wall, and of a plasma bounded by vacuum. All edge effects in the plasma are neglected. The dispersion relations have been written down for a cold plasma only but the straightforward generalization to a warm plasma is indicated and will be carried through in a later report. The neglect of the plasma temperature restricts the dispersion relations of this report to modes that satisfy

$$\frac{|\omega + i\,\nu_e|^2}{k^2 c^2} \gg \frac{K T_e}{m c^2} \quad ,$$

as well as the high frequency criterion. Here ν_e is the plasma electron collision frequency, K is Boltzmann's constant, T_e is the plasma electron temperature, and mc^2 is the electron rest energy. For nonrelativistic plasma thermal energies, the plasma temperature will therefore be of importance only when

$$\nu_e \ll kc \gg \omega + \omega_\beta .$$

CONTENTS

I INTRODUCTION

An analysis of a finite radius beam-plasma system is begun in this report. The main concern of this work will be to determine the effects of plasma and beam temperature upon the stability of the system. The analysis is restricted to modes for which the pinch field of the beam can be ignored. Only the $\ell = 0$ normal mode is considered here. Higher normal modes will be treated in a subsequent report.

The problem is investigated in the framework of the two-mass approximation, which is valid for an extreme relativistic beam. The infinitely long electron beam and plasma are assumed to have uniform densities both inside and outside the beam radius. The further assumption of charge neutrality necessitates a discontinuous jump in the plasma density at the beam radius. (That is, in the unperturbed system, $n_{01} + n_{02} = n_{03}$ for $r < r_0$ and $n_{02} = n_{03}$ for $r > r_0$ where n_{01}, n_{02}, and n_{03} are the densities of the beam electrons, plasma electrons, and plasma ions, respectively, and r_0 is the beam radius. The usual weak beam assumption corresponds to taking $n_{01} \ll n_{02}$ and $n_{02} \simeq n_{03}$ for all r.)

Dispersion relations are derived for an infinite plasma, for a plasma confined by a cylindrical conducting wall, and for a finite radius plasma bounded by vacuum. All edge effects in the plasma, such as the plasma sheath, are neglected.

The current response of both the beam and the plasma are calculated using the Boltzmann equation. The restriction to high-frequency modes is imposed so that the self-magnetic field of the beam can be neglected, leading to great simplification. It will be shown in detail, in the second report of this series, that the neglect of the pinch field imposes the following restraint on the frequency:

$$|\omega - k\,V_0|^2 \gg \omega_\beta^2$$

where V_0 is the drift velocity and ω_β the betatron frequency of the beam. Hence, for example, the "hose instability" in which $|\omega - k\,V_0|^2 \approx \omega_\beta^2$ will not be treated in this work.

II MATHEMATICAL FORMULATION

Each separate particle species is taken to be described by an exact one-particle distribution function, $f^T(\vec{x}, \vec{p}, t)$, depending upon space, relativistic momentum, and time (the superscript, T, will always refer to exact or 'True') which satisfies the collisionless Boltzmann equation

$$\frac{\partial f_\alpha^T}{\partial t} + \vec{V}_\alpha \cdot V_{x_\alpha} f_\alpha^T + \vec{F}_\alpha^T \cdot \nabla_{p_\alpha} f_\alpha^T = 0 \tag{1}$$

where

$$\vec{V}_\alpha = \frac{d\vec{x}_\alpha}{dt} \qquad \vec{P}_\alpha = \gamma m_\alpha \vec{V}_\alpha$$

$$\gamma = \left[1 + p_\alpha^2/m_\alpha^2 c^2 \right]^{1/2} = \left[1 - V_\alpha^2/c^2 \right]^{-1/2}$$

$$\vec{F}_\alpha^T = \frac{d\vec{p}_\alpha}{dt} = e_\alpha \left[\vec{E}^T + \frac{1}{c} \vec{V}_\alpha \times \vec{B}^T \right]$$

and the notation is used such that $\alpha = 1$ for the beam electrons, $\alpha = 2$ and $\alpha = 3$ denotes the plasma electrons and ions, respectively. The charge e is defined by setting $e_{1,2} = -e$ and $e_3 = +e$.

To analyze the stability of the system, the effect of a small perturbation from equilibrium is calculated in the linear approximation. That is, Eq. (1) is linearized by writing

$$f^T = f_0 + f , \tag{2}$$

where f_0 is the equilibrium distribution and f represents the perturbation such that $|f| << |f_0|$. Thus, keeping terms only to first order in perturbed quantities,

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$$\frac{\partial f_\alpha}{\partial t} + \vec{V}_\alpha \cdot \nabla_{x_\alpha} f_\alpha + \vec{F}_{0\alpha} \cdot \nabla_{p_\alpha} f_\alpha = - \vec{V}_\alpha \cdot \nabla_{x_\alpha} f_{0\alpha}$$

$$- \vec{F}_\alpha^{\,T} \cdot \nabla_{p_\alpha} f_{0\alpha} \tag{3}$$

where the exact force has been separated, $\vec{F}^{\,T} = \vec{F}_0 + \vec{F}$, into the force due to the equilibrium fields

$$\vec{F}_{0\alpha} = e_\alpha \left[\vec{E}_0 + \frac{1}{c} \vec{V}_\alpha \times \vec{B}_0 \right]$$

and that due to the perturbations of the fields

$$\vec{F}_\alpha = e_\alpha \left[\vec{E} + \frac{1}{c} \vec{V}_\alpha \times \vec{B} \right] \ .$$

For any function, $g(\vec{x}, \vec{p}, t)$, the total time derivative is defined by

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \vec{V} \cdot \nabla_x g + \vec{F} \cdot \nabla_p g \ . \tag{4}$$

Hence, Eq. (1) is just Liouville's theorem which states that along the exact particle orbit

$$\frac{df_\alpha^T}{dt} = 0 \ . \tag{5}$$

Therefore, along the exact orbit,

$$\frac{df_\alpha}{dt} = - \frac{df_{0\alpha}}{dt} \ . \tag{6}$$

Equations (3) and (6) are obviously equivalent. Hence a formal solution to Eq. (3) is

$$f_\alpha(\vec{x}, \vec{p}, t) - f_\alpha(\vec{x}_0, \vec{p}_0, t_0) = - \int_{t_0}^{t} \left[\vec{V}' \cdot \nabla_x' f_{0\alpha}' + e_\alpha \left\{ \vec{E}^{T'} + \frac{1}{c} \vec{V}' \times \vec{B}^{T'} \right\} \right] \cdot \nabla_p' f_{0\alpha}' dt'$$

(Exact Orbit) $\tag{7}$

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where all quantities in the integrand are evaluated at time t' along the exact orbit, as indicated by the primes. The first order approximation to Eq. (7) for f_α is obtained by integrating along the zero-order orbit (henceforth denoted by ZOO), which is defined by the condition that, along this orbit,

$$\vec{V}_\alpha \cdot \nabla_{x_\alpha} f_{0\alpha} + e_\alpha \left[\vec{E}_0 + \frac{1}{c} \vec{V}_\alpha \times \vec{B}_0 \right] \cdot \nabla_{p_\alpha} f_{0\alpha} = 0 . \tag{8}$$

In the analysis of instabilities it is assumed that all perturbations are infinitesimal at $t = -\infty$. Therefore the basic equation for the perturbation of the Boltzmann distribution is

$$f_\alpha (\vec{x}, \vec{p}, t) = - e_\alpha \int_{-\infty}^{t} \left[\vec{E'} + \frac{1}{c} \vec{V'} \times \vec{B'} \right] \cdot \nabla_p f'_{0\alpha} \, dt' . \tag{9}$$
$$(ZOO)$$

To determine the time development of a disturbance in the system, it is necessary to solve Maxwell's equations, subject to the proper boundary conditions. The current and charge distributions are calculated using the Boltzmann distribution functions. Maxwell's equations for the exact fields are

$$\nabla \cdot \vec{E}^T = 4\pi \rho_Q^T \tag{10a}$$

$$\nabla \times \vec{E}^T = -\frac{1}{c} \frac{\partial \vec{B}^T}{\partial t} \tag{10b}$$

$$\nabla \cdot \vec{B}^T = 0 \tag{10c}$$

$$\nabla \times \vec{B}^T = \frac{4\pi}{c} \vec{J}^T + \frac{1}{c} \frac{\partial \vec{E}^T}{\partial t} . \tag{10d}$$

Let

$$\vec{E}^T = \vec{E}_0 + \vec{E} \tag{11a}$$

$$\vec{B}^T = \vec{B}_0 + \vec{B} \tag{11b}$$

$$\rho_Q^T = \rho_{Q0} + \rho_Q \tag{11c}$$

$$\vec{J}^T = \vec{J}_0 + \vec{J} \tag{11d}$$

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where, as always, the subscript 0 denotes zero order, time-independent, equilibrium quantities and all quantities without subscripts are small perturbations. Inserting Eq. (11) into Eq. (10)

$$\nabla \cdot \vec{E}_0 = 4\pi \rho_{Q0} \tag{12a}$$

$$\nabla \cdot \vec{B}_0 = 0 \tag{12b}$$

$$\nabla \times \vec{B}_0 = \frac{4\pi}{c} \vec{J}_0 \tag{12c}$$

and

$$\nabla \cdot \vec{E} = 4\pi \rho_Q \tag{12d}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \tag{12e}$$

$$\nabla \cdot \vec{B} = 0 \tag{12f}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} . \tag{12g}$$

In terms of the distribution functions, $f_{0\alpha}$ and f_{α},

$$\rho_{Q0} = \sum_\alpha e_\alpha \int d^3p \, f_{0\alpha} \tag{13a}$$

$$\vec{J}_0 = \sum_\alpha e_\alpha \int d^3p \, \vec{V} f_{0\alpha} \tag{13b}$$

$$\rho_Q = \sum_\alpha e_\alpha \int d^3p \, f_\alpha \tag{13c}$$

$$\vec{J} = \sum_\alpha e_\alpha \int d^3p \, \vec{V} f_\alpha \tag{13d}$$

The basic equations for this problem are Eqs. (1), (9), (12), and (13) together with the proper boundary conditions at, for example, the interface between the beam and plasma and the correct regularity conditions at the origin and infinity.

The procedure to be followed is now outlined:

(1) First the self-consistent equilibrium distributions, $f_{0\alpha}$, are found satisfying Eq. (1) with $\frac{\partial f_{0\alpha}}{\partial t} = 0$, Eqs. (12a-c), and Eqs. (13a,b).

(2) Equation (9) is used to calculate the f_{α}'s as functionals of the electric and magnetic field components.

(3) The response of the beam and the plasma to the perturbation is then found by calculating ρ_Q and $\vec{J}$ using Eqs. (13c,d). Therefore ρ_Q and $\vec{J}$ are also found as functionals of $\vec{E}$ and $\vec{B}$.

(4) Inserting ρ_Q and $\vec{J}$ into Eqs. (12d-g) produces, in general, a set of coupled integro-differential equations for the field components.

(5) The dispersion relations describing the propagation of the disturbance are then obtained as "solvability conditions" for the set of equations. That is, if the equations are just ordinary differential equations, the dispersion relations will result when the solutions are fitted across the beam-plasma interface (and across any other boundary that might exist). If the equations are purely algebraic (as they are, for example, when the beam and plasma are each homogeneous and infinite in extent[*1]) or if the equations are integro-differential the dispersion relations arise as requirements such that the set of equations have a non-trivial solution.[2]

A summary of previous work along these same lines for a relativistic beam now follows:

Ascoli—The procedure, as formulated above, is essentially the same as that of G. Ascoli.[3] However, the beam response is calculated by Ascoli by using a simplifying assumption about E (e.g., E_z (r) $\simeq$ Ar within the beam, where A is a constant) and the plasma is considered as a simple ohmic medium. The calculation includes the effect of the self-field of the beam on the beam itself in the approximation that the longitudinal mass of the beam particles is

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*References are listed at the end of the report.

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infinite. Both beam and plasma are taken as having zero temperature. (A later report by Ascoli,[4] which incorporates thermal effects, is a heuristic treatment and is not a systematic integration of the Boltzmann equation.)

BWR—Bludman, Watson, and Rosenbluth[1] consider a uniform beam and uniform plasma, both of infinite radius. Therefore, a complete Fourier analysis of the Boltzmann equation is possible and an orbit integration is unnecessary. The effect of the magnetic field due to the beam is neglected and the plasma is taken as essentially ohmic, collisions in the plasma being taken into account by introducing a constant, velocity-independent phenomenological collision frequency. Different longitudinal and transverse temperatures are assumed for the beam. The electrostatic instabilities are found and growth rates computed.

FGWWR—Frieman, et al.,[5] is a continuation of BWR, where the beam is taken to be of finite radius, of nonuniform density, and having zero temperature. All problems regarding the solution of the Boltzmann equation are circumvented by assuming that the current responses of the beam and plasma are given by

$$- 4\pi i\omega \, \vec{J}^P = \omega_P^2 \, \vec{E} \qquad (14a)$$

$$- 4\pi i\omega' \, \vec{J}^{B'} = \omega_\perp'^2 \, \vec{E}' \qquad (14b)$$

where ω_P and $\omega_\perp$ are the usual plasma and transverse beam frequencies and the primed quantities are measured in the rest frame of the beam. The problem is examined only in the case of very specific approximations, which may severely limit the applicability of the results. Essentially the same instabilities and growth rates are found as were found by BWR, though the dispersion relations have a completely different formal structure.

Mjolsness—R. Mjolsness[2] considers a finite, zero-temperature beam of uniform density penetrating an ohmic plasma in the approximation of infinite longitudinal mass for the beam particles. Unlike Ascoli[3] above, Mjolsness derives the exact integral equations for the fields for arbitrary normal mode.

The resulting very complicated dispersion relations come about as a solvability condition for the set of integral equations.

This report is a direct extension of BWR to include a finite beam radius and nonzero plasma temperature. Equivalently it can be considered an extension of the work of Mjolsness to include the effect of temperature upon the high-frequency stability of the system.

The procedure of this paper is, in principle, general enough so that all types of instabilities (streaming, electrostatic, hose, etc.) can be treated on an equal footing. However the restriction to frequencies such that $|\omega - k\,V_0|^2 >> \omega_\beta^2$ limits the present analysis to instabilities that are most closely related to the streaming instabilities of the hydrodynamic models.

III CALCULATION OF THE PERTURBED DISTRIBUTION FUNCTIONS

In order to use Eq. (9) to calculate the f_α's, it is first necessary to specify the zero-order orbits and the equilibrium distributions. It is assumed here that there are no external fields and all particle densities are uniform inside and outside the beam. Charge neutrality further requires a discontinuous jump in the plasma density at the beam radius. The self-magnetic field of the beam will be neglected. The effect of ignoring the pinch field is expected to be negligible for Doppler-shifted frequencies much greater than the betatron frequency. That is, the results of this report are valid for modes that satisfy

$$|\Omega|^2 \gg \omega_\beta^2$$

where

$$\Omega = \omega - k V_0 \; , \quad \omega_\beta^2 = 2 \frac{V_0^2}{c^2} \omega_\perp^2 \simeq \frac{1}{2} \omega_\perp^2 \; , \quad \omega_\perp^2 = \frac{4\pi e^2 n_{01}}{\gamma_0 m}$$

In the above, V_0 is the drift velocity of the beam, m is the electron mass, n_{01} is the number density of the beam and $\gamma_0 = [1 - V_0^2/c^2]^{-1/2}$.

The over-all effect of the above assumptions is to make the equilibrium configuration field-free. The relativistic equilibrium distribution for a beam with drift momentum $\vec{p}_0$ and temperature T is[6]

$$f_0 = A e^{-\frac{mc^2}{KT} \left[\gamma - \frac{\vec{V}_0 \cdot \vec{p}}{mc^2} \right]}$$

(15)

where

$$\gamma = [1 + p^2/m^2c^2]^{1/2}$$
$$\vec{V}_0 = \vec{p}_0/\gamma_0 m$$
$$\gamma_0 = [1 - V_0^2/c^2]^{-1/2} \; .$$

K is the Boltzmann constant and A is a normalization constant determined from

$$\int d^3p \, f_0 \ = \ n_0 \tag{16}$$

where n_0 is the number density of beam particles. Assuming with BWR[1] that a "quasi-steady state" distribution about the drift motion will obtain such that $\vec{p} = \vec{p}_0 + \vec{q}$, where $|\vec{q}| \ll |\vec{p}_0|$, then, to second order in $|\vec{q}|$ and generalizing to separate temperatures parallel and transverse to the drift velocity, the equilibrium distribution for the beam is

$$f_{01} \ = \ \frac{n_{01}}{(2\pi\gamma_0 m\theta_\perp)(2\pi\gamma_0^3 m\theta_{||})^{1/2}} \, e^{-\frac{1}{2}\left[\frac{q_\perp^2}{\gamma_0 m\theta_\perp} + \frac{q_{||}^2}{\gamma_0^3 m\theta_{||}}\right]} \ . \tag{17}$$

In Eq. (17) $\theta_\perp$ and $\theta_{||}$ are the transverse and longitudinal temperatures (multiplied by K), respectively, and $\vec{q} = \vec{q}_{||} + \vec{q}_\perp$, where $\vec{q}_{||} = (\vec{q} \cdot \vec{p}_0)\vec{p}_0/p_0^2$. The transition from Eq. (15) to Eq. (17) has effectively introduced the two-mass approximation[7] for the random motion

$$\vec{q}_{||} \ = \ \gamma_0^3 \, m \, \vec{v}_{||} \tag{18a}$$

$$\vec{q}_\perp \ = \ \gamma_0 \, m \, \vec{v}_\perp \ . \tag{18b}$$

In terms of the random velocities

$$f_{01} \ = \ \frac{n_{01}}{(2\pi\gamma_0 m\theta_\perp)(2\pi\gamma_0^3 m\theta_{||})^{1/2}} \, e^{-\frac{1}{2}\left[\frac{\gamma_0 m}{\theta_\perp} v_\perp^2 + \frac{\gamma_0^3 m}{\theta_{||}} v_{||}^2\right]} \tag{19}$$

and

$$\int d^3 v \, f_{01} \ = \ \frac{n_{01}}{\gamma_0^5 m^3} \ . \tag{20}$$

Equation (17), or equivalently, Eq. (19) is assumed to hold only for $r < r_0$ where r_0 is the radius of the beam. For $r > r_0$, f_{01} is zero. However a uniform density, a uniform transverse temperature distribution and a sharp boundary are difficult to reconcile in the equilibrium situation unless a confining force is present (as in actuality there is, namely the pinch field). Therefore since the equilibrium

configuration was assumed to be field-free, this difficulty will be circumvented by computing the perturbed distribution function, f_1, only to zero order in $\theta_\perp$. The plasma is assumed to be nonrelativistic, to have no drift velocity, and to have separate ion and electron temperature distributions. The equilibrium distributions are taken as Maxwellian:

$$f_{02} = \frac{n_{02}}{(2\pi m K T_e)^{3/2}} \, e^{-\frac{m}{2K T_e} v^2} \tag{21a}$$

and

$$f_{03} = \frac{n_{03}}{(2\pi M K T_i)^{3/2}} \, e^{-\frac{M}{2K T_i} v^2} \tag{21b}$$

where T_e and T_i are the electron and ion temperatures, respectively, M is the ionic mass, and, due to the assumption of a field-free equilibrium configuration, for $r < r_0$, $n_{01} + n_{02} = n_{03}$ and for $r > r_0$, $n_{02} = n_{03}$.

The zero-order orbits for $t' < t$ are determined according to the conditions that, when $t' = t$,

$$\vec{V}' \equiv \frac{d\vec{x}'}{dt'} = (V_0 + v_z) \, \hat{z} + \vec{v}_\perp$$

$$\vec{x}' = \vec{x} = z \, \hat{z} + \vec{r}$$

where the z direction has been defined to be in the direction of the drift velocity. In equilibrium

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \tag{22a}$$

and since in the two-mass approximation (i.e., the momentum expanded to first order in the random velocities)

$$\vec{p} \simeq \gamma_0 m \vec{V}_0 + \gamma_0 m \vec{v}_\perp + \gamma_0^3 m \vec{v}_z \, , \tag{22b}$$

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then

$$\frac{d\vec{v}_\perp'}{dt'} = 0 \qquad\qquad \frac{dV_z'}{dt'} = 0 \ . \qquad (22c)$$

Hence

$$\frac{dx'}{dt'} = v_x \qquad \frac{dy'}{dt'} = v_y \qquad \frac{dz'}{dt'} = V_0 + v_z \qquad (23)$$

and so, the zero-order orbits are ($V_0 = 0$ for the plasma)

$$x' = x - v_x (t - t')$$

$$y' = y - v_y (t - t') \qquad (24)$$

$$z' = z - (V_0 + v_z)(t - t') \ .$$

Therefore the neglect of the pinch field allows the use of simple straight-line orbits for all particles. In the second report of this series, more accurate orbits (including the betatron oscillation) will be used to show that identical results are obtained in the limit $|\Omega|^2 \gg \omega_\beta^2$ as are found by using straight-line orbits.

The calculation of f_1 will now be shown in some detail. f_2 and f_3 can be obtained from f_1 in an obvious manner.

$$f_1(\vec{x}, \vec{p}, t) = e \int_{-\infty}^{t} \left[\vec{E}(\vec{x}', t') + \frac{1}{c} \vec{V}_1' \times \vec{B}(\vec{x}', t') \right] \cdot \nabla_{p_1'} f_{01}' \, dt' \ . \qquad (25)$$
$$(ZOO)$$

Let

$$\vec{E}(\vec{x}, t) = \vec{E}(\vec{r}) \, e^{i(kz - \omega t)} \ . \qquad (26)$$

Then, using Eq. (12e),

$$\vec{B}(\vec{r}) = \frac{c}{i\omega} (\nabla_\perp + ik\hat{z}) \times \vec{E}(\vec{r}) \qquad (27)$$

where

$$\nabla_\perp = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \ .$$

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Also

$$\nabla_{p_1} f_{01} = - \left[\frac{\vec{v}_\perp}{\theta_\perp} + \frac{\vec{v}_z}{\theta_\parallel} \right] f_{01} .$$ (28)

Thus

$$f_1(\vec{x}, p, t) = - e \int_{-\infty}^{t} \left[\vec{\tilde{E}}(\vec{r}') + \frac{1}{i\omega} \vec{V}_1' \times (\nabla_\perp' + ik\,\hat{z}) \times \vec{\tilde{E}}(\vec{r}') \right] e^{i(kz' - \omega t')}$$ (ZOO)

(29)

$$\cdot \left[\frac{\vec{v}_\perp'}{\theta_\perp} + \frac{\vec{v}_z'}{\theta_\parallel} \right] f_{01}' \, dt' .$$

Using Eq. (24)

$$e^{i(kz' - \omega t')} = e^{i(kz - \omega t)} e^{i\left[\Omega - kv_z\right](t - t')}$$ (30)

where

$$\Omega = \omega - kV_0 + i\,\nu_B \quad (\nu_B > 0) .$$

The inclusion of ν_B defines the assymptotic conditions at $t' = -\infty$ so that the integrals over t' converge. If for the plasma ν_e and ν_i, where ν_e and ν_i are the electron and ion collision frequencies, respectively, are retained after the integration over t' is performed, it will be apparent that, in the cold plasma limit, they are practically identical (for $n_{02} \simeq n_{03}$ and $M \gg m$ they are exactly the same) with the constant, velocity-independent collision frequencies as usually introduced when working in the framework of the hydrodynamical approximation. It will be assumed that $\nu_B \simeq 0$. (This is a good approximation unless $\nu_B \approx |\omega - kV_0|$. Consequences of $\nu_B \neq 0$ will be discussed in the second report of this series.) ν_e and ν_i will be retained as a measure of the collisions taking place in the plasma.

Thus, letting $\tau = t - t'$,

$$f_1(\vec{x}, \vec{p}, t) = -e\, e^{i(kz - \omega t)} \int_0^\infty \left[1 + \frac{\vec{V}_0 + \vec{v}}{i\omega} \times (\nabla_\perp + ik\,\hat{z}) \times \right] \vec{\vec{E}}\,(x - v_x\tau,\ y - v_y\tau)$$

$$\cdot\, e^{i\left[\Omega - kv_z\right]\tau} \left[\frac{\vec{v}_\perp}{\theta_\perp} + \frac{\vec{v}_z}{\theta_{\|}} \right] f_{01}\, d\tau \ . \tag{31}$$

Let

$$\vec{\vec{G}}(x', y') = \left[1 + \frac{\vec{V}_0 + \vec{v}}{i\omega} \times (\nabla_\perp + ik\,\hat{z}) \times \right] \vec{\vec{E}}\,(x - v_x\tau,\ y - v_y\tau) \ . \tag{32a}$$

Then, expanding in powers of $v_x\tau$ and $v_y\tau$,

$$\vec{\vec{G}}(x', y') = \vec{\vec{G}}(x, y) - \left[v_x \frac{\partial}{\partial x'} + v_y \frac{\partial}{\partial y'} \right] \vec{\vec{G}}(x', y')\Big|_{\substack{x' = x \\ y' = y}} \tau + \dots \tag{32b}$$

Using Eq. (24) and letting $\vec{r} = x\,\hat{x} + y\,\hat{y}$,

$$\vec{\vec{G}}(\vec{r}') = \vec{\vec{G}}(\vec{r}) - (\vec{v}_\perp \cdot \nabla_\perp)\vec{\vec{G}}(\vec{r})\,\tau + \frac{1}{2}(\vec{v}_\perp \cdot \nabla_\perp)^2\, \vec{\vec{G}}(r)\,\tau^2 + \dots \ . \tag{32c}$$

The integration over τ is straightforward using Eq. (32c). The result is written in operator form as

$$\int_0^\infty \vec{\vec{G}}(\vec{r}')\, e^{i\left[\Omega - kv_z\right]\tau}\,d\tau = \frac{i}{\Omega - kv_z + i(\vec{v}_\perp \cdot \nabla_\perp)}\, \vec{\vec{G}}(\vec{r}) \tag{33}$$

(the right-hand side is defined by the expansion of $(1 + x)^{-1}$ for $|x| < 1$).

Let

$$f_\alpha(\vec{x}, \vec{p}, t) = f_\alpha(\vec{r}, \vec{p})\, e^{i(kz - \omega t)} \ . \tag{34}$$

Therefore, the results for the perturbed distribution functions are

$$f_1(\vec{r}, \vec{p}) = -\frac{i\, e}{\Omega - kv_z + i(\vec{v}_\perp \cdot \nabla_\perp)} \left[\vec{\vec{E}}(\vec{r}) + \frac{\vec{V}_0 + \vec{v}}{i\omega} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{\vec{E}}(\vec{r}) \right]$$

$$\cdot \left[\frac{\vec{v}_\perp}{\theta_\perp} + \frac{\vec{v}_z}{\theta_{\|}} \right] f_{01} \tag{35a}$$

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$$f_2(\vec{r}, \vec{p}) = - \frac{i\,e}{\omega + i\,\nu_e - k\,v_z + i\,(\vec{v}_\perp \cdot \vec{\nabla}_\perp)} \left[\vec{E}(\vec{r}) + \frac{\vec{v}}{i\omega} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{E}(\vec{r}) \right] \cdot \frac{\vec{v}}{K\,T_e} f_{02}$$

<div align="right">(35b)</div>

and

$$f_3(\vec{r}, \vec{p}) = + \frac{i\,e}{\omega + i\,\nu_i - k\,v_z + i\,(\vec{v}_\perp \cdot \vec{\nabla}_\perp)} \left[\vec{E}(\vec{r}) + \frac{\vec{v}}{i\omega} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{E}(\vec{r}) \right] \cdot \frac{\vec{v}}{K\,T_i} f_{03} \;.$$

<div align="right">(35c)</div>

IV. THE BEAM AND PLASMA CURRENT RESPONSE

Using Eqs. (13a–d), (17), and (21a,b) the beam and plasma currents are found to be given by

$$\vec{J}_0^B = - e\, n_{01}\, \vec{V}_0 \tag{36a}$$

$$\vec{J}^B(\vec{x},\vec{p},t) = - e\gamma_0^5 m^3 \int d^3v\, (\vec{V}_0 + \vec{v})\, f_1(\vec{x},\vec{p},t) \tag{36b}$$

$$\vec{J}_0^P = 0 \tag{36c}$$

$$\vec{J}^P(\vec{x},\vec{p},t) = - e \int d^3v\, \vec{v}[m^3 f_2(\vec{x},\vec{p},t) + M^3 f_3(\vec{x},\vec{p},t)]\ . \tag{36d}$$

Then, using Eq. (35a–c),

$$\vec{J}^B(\vec{r},\vec{p}) = e^2\gamma_0^5 m^3 \int d^3v\, (\vec{V}_0 + \vec{v})\, \frac{i}{\Omega - k\,v_z + i\,\vec{V}_\perp \cdot \nabla_\perp}\left[\vec{E}(\vec{r}) + \frac{\vec{V}_0 + \vec{v}}{i\omega}\, x\, (\nabla_\perp + ik\,\hat{z})\right.$$
$$\left. x\ \vec{E}(\vec{r})\right] \cdot \left[\frac{\vec{v}_\perp}{\theta_\perp} + \frac{\vec{v}_z}{\theta_{||}}\right] f_{01} \tag{37}$$

and

$$\vec{J}^P(\vec{r},\vec{p}) = e^2 \int d^3v\, \vec{v}\, \frac{i}{\omega + i\,\nu_p - k\,v_z + i\,\vec{V}_\perp \cdot \nabla_\perp}\left[\vec{E}(\vec{r}) + \frac{\vec{v}}{i\omega}\, x\, (\nabla_\perp + ik\,\hat{z})\, x\, \vec{E}(\vec{r})\right]$$
$$\cdot \vec{v}\left[\frac{m^3}{K\,T_e} f_{02} + \frac{M^3}{K\,T_i} f_{03}\right]\ . \tag{38}$$

In Eq. (38), ν_P represents either ν_e or ν_i, this differentiation being reinstated in the final result given for $\vec{J}$ (see p. 23).

For the reasons cited following Eq. (20), the beam current will be calculated to arbitrary order in $\theta_{||}$ but only to zero order in $\theta_\perp$. Also, it is obvious from the form of the denominator in Eq. (38) that the major effect of plasma temperature is due to the thermal motion in the z direction. Hence for simplicity, the plasma current will be computed only up to the first nonvanishing contribution

of the transverse thermal motion. That is, terms of order $\frac{KT_e}{mc^2}$ and $\frac{KT_i}{Mc^2}$ are being neglected relative to unity but all terms of order $\frac{|\omega + i\,\nu_e|}{k\,U_e}$ and $\frac{|\omega + i\,\nu_i|}{k\,U_i}$, for example, are retained where $KT_e = m\,U_e^2$ and $KT_i = M\,U_i^2$.

$$\vec{J}_\perp^B \simeq e^2 \gamma_0^5 m^3 \int d^3v \, \frac{i\,\vec{v}_\perp}{\Omega - k\,v_z} \left[\vec{\vec{E}} + \frac{V_0 + v_z}{i\omega} \, \hat{z} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{\vec{E}} \right] \cdot \frac{\vec{v}_\perp}{\theta_\perp} f_{01} \quad (39a)$$

$$J_z^B \simeq e^2 \gamma_0^5 m^3 \int d^3v \, \frac{i(V_0 + v_z)}{\Omega - k\,v_z} \left\{ \left[\vec{\vec{E}} + \frac{V_0 + v_z}{i\omega} \, \hat{z} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{\vec{E}} \right] \cdot \frac{\vec{v}_z}{\theta_{\|}} \right.$$

$$\left. + \left[-\frac{i(\vec{v}_\perp \cdot \nabla_\perp)}{\Omega - k\,v_z} \, \vec{\vec{E}} - \frac{V_0 + v_z}{\omega(\Omega - k\,v_z)} (\vec{v}_\perp \cdot \nabla_\perp) \, \hat{z} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{\vec{E}} + \frac{\vec{v}_\perp}{i\omega} \right. \quad (39b)$$

$$\left. \times (\nabla_\perp + ik\,\hat{z}) \times \vec{\vec{E}} \right] \cdot \frac{\vec{v}_\perp}{\theta_\perp} \bigg\} f_{01}$$

and

$$\vec{J}_\perp^P \simeq e^2 \int d^3v \, \frac{i\,\vec{v}_\perp}{\omega + i\,\nu_P - k\,v_z} \left[\left(1 - \frac{i\,\vec{v}_\perp \cdot \nabla_\perp}{\omega + i\,\nu_P - k\,v_z} \right) \left(\vec{\vec{E}} + \frac{\vec{v}}{i\omega} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{\vec{E}} \right) \right]$$

$$(39c)$$

$$\cdot \vec{v} \left[\frac{m^3}{KT_e} f_{02} + \frac{M^3}{KT_i} f_{03} \right]$$

$$J_z^P \simeq e^2 \int d^3v \, \frac{i\,v_z}{\omega + i\,\nu_P - k\,v_z} \left[\left(1 - \frac{i\,\vec{v}_\perp \cdot \nabla_\perp}{\omega + i\,\nu_P - k\,v_z} - \frac{(\vec{v}_\perp \cdot \nabla_\perp)^2}{(\omega + i\,\nu_P - k\,v_z)^2} \right) \right.$$

$$(39d)$$

$$\left. \left(\vec{\vec{E}} + \frac{\vec{v}}{i\omega} \times (\nabla_\perp + ik\,\hat{z}) \times \vec{\vec{E}} \right) \right] \cdot \vec{v} \left[\frac{m^3}{KT_e} f_{02} + \frac{M^3}{KT_i} f_{03} \right] .$$

(The vector operations in the above formulas are to be carried out first within the square bracket and stepwise from right to left.)

The vector identities needed for this computation are listed in the appendix, expressed in cylindrical coordinates. The integration over $\vec{v}$ is done using the following formulas, which are easily verified:

$$\int d^3v \, f_{01} \;=\; \frac{n_{01}}{\gamma_0^5 \, m^3}$$

$$\int d^3v \, f_{02} \;=\; \frac{n_{02}}{m^3}$$

$$\int d^3v \, v_r^2 \, f_{01} \;=\; \int d^3v \, v_\theta^2 \, f_{01} \;=\; \frac{\theta_\perp}{\gamma_0 m} \, \frac{n_{01}}{\gamma_0^5 \, m^3}$$

$$\int d^3v \, v_z^2 \, f_{01} \;=\; \frac{\theta_\|}{\gamma_0^3 \, m} \, \frac{n_{01}}{\gamma_0^5 \, m^3} \qquad (40)$$

$$\int d^3v \, v_i^2 \, f_{02} \;=\; \frac{K \, T_e}{m} \, \frac{n_{02}}{m^3}$$

$$\int d^3v \, v_i^2 \, v_j^2 \, f_{02} \;=\; \left(\frac{K \, T_e}{m} \right)^2 \frac{n_{02}}{m^3} \quad (i \neq j)$$

$$\int d^3v \, (v_i^2)^2 \, f_{02} \;=\; 3 \left(\frac{K \, T_e}{m} \right)^2 \frac{n_{02}}{m^3} \, .$$

Also

$$\int d^3v \, v_i^2 \, F \;=\; \frac{\theta_i}{m_i} \, \int d^3v \, F \qquad (41)$$

if F depends upon v_i only in the form $e^{-\frac{1}{2} \, \frac{m_i}{\theta_i} \, v_i^2}$.

Defining $E_z' \;=\; \dfrac{d \, E_z}{dr}$, $E_z'' \;=\; \dfrac{d^2 E_z}{dr^2}$, etc., then the result for the beam current is

$$- 4\pi i \omega \, J_r^B \;=\; B_1 \, E_r \;+\; B_3 \, E_z'$$

$$- 4\pi i \omega \, J_\theta^B \;=\; B_1 \, E_\theta \qquad (42)$$

$$- 4\pi i \omega \, J_z^B \;=\; B_2 \, E_z \;+\; B_3\left(E_r' + \frac{1}{r} \, E \right) \;+\; B_4 \left(E_z'' + \frac{1}{r} \, E_z' \right)$$

where

$$B_1 \;=\; \omega_\perp^2$$

$$B_2 \;=\; \omega_\|^2 \, \frac{\gamma_0^5 \, m^3}{n_{01}} \, \frac{\gamma_0^3 \, m}{\theta_\|} \, \omega \int d^3v \, \frac{(V_0 + v_z) v_z}{\Omega - k v_z} \, f_{01}$$

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$$B_3 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} i \int d^3v \frac{V_0 + v_z}{\Omega - k v_z} f_{01}$$

$$B_4 = -\omega_\perp^2 \frac{\gamma_0^5 m^3}{n_{01}} \int d^3v \frac{(V_0 + v_z)^2}{(\Omega - k v_z)^2} f_{01}$$

and

$$\omega_\perp^2 = \frac{4\pi e^2 n_{01}}{\gamma_0 m}$$

$$\omega_\parallel^2 = \frac{4\pi e^2 n_{01}}{\gamma_0^3 m} .$$

The plasma current is

$$- 4\pi i \omega J_r^P = (P_1 + P_2) E_r + P_3 E_z'$$

$$- 4\pi i \omega J_\theta^P = (P_1 + P_2) E_\theta \tag{43}$$

$$- 4\pi i \omega J_z^P = (P_1 + P_4) E_z + P_3 \left(E_r' + \frac{1}{r} E_r \right) + P_5 \left(E_z'' + \frac{1}{r} E_z' - \frac{ik}{r} E_r \right) + P_6 E_z''$$

where

$$P_1 = \omega_P^2$$

$$P_2 = \int d^3v \frac{k v_z}{\omega + i \nu_P - k v_z} f_{23}$$

$$P_3 = -i(\omega + i \nu_P) \int d^3v \frac{v_z}{(\omega + i \nu_P - k v_z)^2} f_{23}$$

$$P_4 = \int d^3v \frac{k v_z^3}{\omega + i \nu_P - k v_z} \left[\omega_e^2 \frac{m^3}{n_{02}} \frac{m}{K T_e} f_{02} + \omega_i^2 \frac{M^3}{n_{03}} \frac{M}{K T_i} f_{03} \right]$$

$$P_5 = - \int d^3v \frac{v_z^2}{(\omega + i \nu_P - k v_z)^2} f_{23}$$

$$P_6 = - \int d^3v \frac{k v_z^3}{(\omega + i \nu_P - k v_z)^3} f_{23} .$$

(Note that $P_2 = ik P_3 + k^2 P_5$.)

Here

$$f_{23} = \omega_e^2 \frac{m^3}{n_{02}} f_{02} + \omega_i^2 \frac{M^3}{n_{03}} f_{03}$$

and

$$\omega_P^2 = \omega_e^2 + \omega_i^2$$

where

$$\omega_e^2 = \frac{4\pi e^2 n_{02}}{m} \frac{\omega}{\omega + i\nu_e}$$

and

$$\omega_i^2 = \frac{4\pi e^2 n_{03}}{M} \frac{\omega}{\omega + i\nu_i} \ .$$

The total perturbed current can be written as

$$- 4\pi i\omega\, J_r = \sigma_1^2\, E_r + \sigma_2^2\, \frac{1}{ik}\, E_z'$$

$$- 4\pi i\omega\, J_\theta = \sigma_1^2\, E_\theta \tag{44}$$

$$- 4\pi i\omega\, J_z = \sigma_3^2\, E_z + \sigma_2^2\, \frac{1}{ik}\, E_r' + \sigma_4^2\, \frac{1}{ik}\, \frac{1}{r}\, E_r - \sigma_5^2\, \frac{1}{k^2} \left(E_z'' + \frac{1}{r}\, E_z' \right) - \sigma_6^2\, \frac{1}{k^2}\, E_z''$$

where

$$\sigma_1^2 = B_1 + P_1 + P_2$$

$$\sigma_2^2 = ik\,(B_3 + P_3)$$

$$\sigma_3^2 = B_2 + P_1 + P_4$$

$$\sigma_4^2 = ik\, B_3 + P_2$$

$$\sigma_5^2 = -k^2\,(B_4 + P_5)$$

$$\sigma_6^2 = -k^2\, P_6 \ .$$

In terms of the usual dispersion integral*

$$Z_j(s_j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-\frac{1}{2}t^2}}{t - s_j - i\epsilon} = \left[i\sqrt{\frac{\pi}{2}} - \int_0^{s_j} dt\, e^{\frac{1}{2}t^2} \right] e^{-\frac{1}{2}s_j^2} \tag{45}$$

where

$$s_B = \frac{\Omega}{k\,U_B} \qquad \text{for the beam}$$

$$s_e = \frac{\omega + i\,\nu_e}{k\,U_e} \qquad \text{for the plasma electrons}$$

$$s_i = \frac{\omega + i\,\nu_i}{k\,U_i} \qquad \text{for the plasma ions}$$

and $\quad \theta_{\parallel} = \gamma_0^3\, m\, U_B^2 \qquad K\,T_e = m\,U_e^2 \qquad K\,T_i = M\,U_i^2$,

then

$$\sigma_1^2 = \omega_{\perp}^2 + \omega_P^2 - \omega_e^2\left[1 + s_e Z_e\right] - \omega_i^2\left[1 + s_i Z_i\right]$$

$$\sigma_2^2 = -\omega_{\perp}^2\left[1 + \frac{\omega}{k\,U_B}Z_B\right] + \omega_e^2 s_e\left[Z_e + s_e Z_e'\right] + \omega_i^2 s_i\left[Z_i + s_i Z_i'\right]$$

$$\sigma_3^2 = -\omega_{\parallel}^2 \frac{\omega^2}{k^2 U_B^2}\left[1 + s_B Z_B\right] - \omega_e^2 s_e^2\left[1 + s_e Z_e\right] - \omega_i^2 s_i^2\left[1 + s_i Z_i\right]$$

$$\sigma_4^2 = -\omega_{\perp}^2\left[1 + \frac{\omega}{k\,U_B}Z_B\right] - \omega_e^2\left[1 + s_e Z_e\right] - \omega_i^2\left[1 + s_i Z_i\right]$$

$$\sigma_5^2 = \omega_{\perp}^2\left[1 + \frac{2\omega}{k\,U_B}Z_B + \frac{\omega^2}{k^2 U_B^2}Z_B'\right] + \omega_e^2\left[1 + 2 s_e Z_e + s_e^2 Z_e'\right]$$

$$\qquad + \omega_i^2\left[1 + 2 s_i Z_i + s_i^2 Z_i'\right]$$

$$\sigma_6^2 = -\omega_e^2\left[1 + 3 s_e Z_e + 3 s_e^2 Z_e' + \frac{1}{2} s_e^3 Z_e''\right]$$

$$\qquad - \omega_i^2\left[1 + 3 s_i Z_i + 3 s_i^2 Z_i' + \frac{1}{2} s_i^3 Z_i''\right] .$$

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*For discussions of $Z(s)$, see Refs. 4, 8, and 9.

The expansions for $Z(s)$ are, for $|s| > 1$ ("low" temperatures)

$$Z(s) = a \, i \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}s^2} - \frac{1}{s} - \frac{1}{s^3} - \frac{3}{s^5} - \dots \qquad (46a)$$

where

$$a = \begin{cases} 2 & \text{Im } s < 0 \\ 1 & \text{Im } s = 0 \\ 0 & \text{Im } s > 0 \end{cases}$$

and for $|s| < 1$ ("high" temperatures)

$$Z(s) = i \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}s^2} - s + \frac{1}{3}s^3 - \frac{1}{15}s^5 + \frac{1}{105}s^7 + \dots \qquad (46b)$$

[It is of interest to note that since the time dependence has been defined as $e^{-i\omega t}$ then an unstable mode has Im $\omega > 0$ which implies Im $s > 0$. Therefore for an instability $a = 0$ in Eq. (46a). However the imaginary term in (46b) is always present.]

Applying Eq. (46a,b), the following is obtained for ZERO PLASMA TEMPERATURE:

$$\sigma_1^2 = \omega_\perp^2 + \omega_P^2$$

$$\sigma_2^2 = -\omega_\perp^2 \left[1 + \frac{\omega}{k U_B} Z_P \right]$$

$$\sigma_3^2 = \omega_P^2 - \omega_\parallel^2 \frac{\omega^2}{k^2 U_B^2} \left[1 + \frac{\Omega}{k U_B} Z_B \right] \qquad (47a)$$

$$\sigma_4^2 = -\omega_\perp^2 \left[1 + \frac{\omega}{k U_B} Z_B \right]$$

$$\sigma_5^2 = \omega_\perp^2 \left[1 + 2 \frac{\omega}{k U_B} Z_B + \frac{\omega^2}{k^2 U_B^2} Z_B' \right]$$

$$\sigma_6^2 = 0.$$

And for ZERO BEAM TEMPERATURE AND ZERO PLASMA TEMPERATURE:

$$\sigma_1^2 = \omega_\perp^2 + \omega_P^2$$

$$\sigma_2^2 = \omega_\perp^2 \frac{k V_0}{\Omega}$$

$$\sigma_3^2 = \omega_P^2 + \omega_\parallel^2 \frac{\omega^2}{\Omega^2}$$

$$\sigma_4^2 = \omega_\perp^2 \frac{k V_0}{\Omega}$$
(47b)

$$\sigma_5^2 = \omega_\perp^2 \frac{k^2 V_0^2}{\Omega^2}$$

$$\sigma_6^2 = 0 .$$

It is to be especially noted that the zero beam temperature and zero plasma temperature limit agrees with the hypothesis of FGWWR, Eqs. (14a,b). Or, equivalently, the statement can be made that, in the limit of zero temperature, the methods of this paper reduce to the usual hydrodynamical treatment.

V MAXWELL'S EQUATIONS

The basic equation for the electric field is

$$\nabla \times (\nabla \times \vec{\tilde{E}}) + \frac{1}{c^2} \frac{\partial^2 \vec{\tilde{E}}}{\partial t^2} = -\frac{4\pi}{c^2} \frac{\partial \vec{\tilde{J}}}{\partial t} \ . \tag{48a}$$

The magnetic field is obtained from

$$\frac{\partial \vec{\tilde{B}}}{\partial t} = -c \, \nabla \times \vec{\tilde{E}} \ . \tag{48b}$$

Defining, as before,

$$\vec{\tilde{E}}(\vec{x}, t) = \vec{\tilde{E}}(\vec{r}) \, e^{i(kz - \omega t)} \tag{49a}$$

then

$$-(\nabla_\perp + ik\hat{z}) \times \left[(\nabla_\perp + ik\hat{z}) \times \vec{\tilde{E}}(\vec{r}) \right] + \frac{\omega^2}{c^2} \vec{\tilde{E}}(\vec{r}) = -\frac{4\pi i\omega}{c^2} \vec{\tilde{J}}(\vec{r}) \tag{49b}$$

and

$$\vec{\tilde{B}}(\vec{r}) = \frac{c}{i\omega} (\nabla_\perp + ik\hat{z}) \times \vec{\tilde{E}}(\vec{r}) \ . \tag{49c}$$

Assuming a normal mode expansion

$$\vec{\tilde{E}}(\vec{r}) = \sum_{\ell=0}^{\infty} \vec{\tilde{E}}_\ell(r) \, e^{i\ell\theta} \tag{50}$$

then Eqs. (49a,b) become (suppressing the subscript 'ℓ')

$$E_z'' + \frac{1}{r} E_z' + \left[\frac{\omega^2}{c^2} - \frac{\ell^2}{r^2} \right] E_z = -\frac{4\pi i\omega}{c^2} J_z + ik \left[E_r' + \frac{1}{r} E_r \right] - \frac{\ell k}{r} E_\theta \tag{51a}$$

$$E_\theta'' + \frac{1}{r} E_\theta' + \left[\frac{\omega^2 - k^2 c^2}{c^2} - \frac{1}{r^2} \right] E_\theta = -\frac{4\pi i\omega}{c^2} J_\theta + \frac{i\ell}{r} \left[E_r' - \frac{1}{r} E_r + ik E_z \right] \tag{51b}$$

$$\left[\frac{\omega^2 - k^2 c^2}{c^2} - \frac{\ell^2}{r^2} \right] E_r = -\frac{4\pi i \omega}{c^2} J_r + ik E_z' + \frac{i\ell}{r} \left[E_\theta' + \frac{1}{r} E_\theta \right] \qquad (51c)$$

$$\frac{i\omega}{c} \vec{B} = \left[\frac{i\ell}{r} E_z - ik E_\theta \right] \hat{r} + \left[ik E_r - E_z' \right] \hat{\theta} + \left[E_\theta' + \frac{1}{r} E_\theta - \frac{i\ell}{r} E_r \right] \hat{z} \quad . \qquad (51d)$$

For the rest of this paper attention will be restricted to the $\ell = 0$ mode. The equations are

$$E_z'' + \frac{1}{r} E_z' + \frac{\omega^2}{c^2} E_z = -\frac{4\pi i \omega}{c^2} J_z + ik \left[E_r' + \frac{1}{r} E_r \right] \qquad (52a)$$

$$E_\theta'' + \frac{1}{r} E_\theta' + \left[\frac{\omega^2 - k^2 c^2}{c^2} - \frac{1}{r^2} \right] E_\theta = -\frac{4\pi i \omega}{c^2} J_\theta \qquad (52b)$$

$$\left[\frac{\omega^2 - k^2 c^2}{c^2} \right] E_r = -\frac{4\pi i \omega}{c^2} J_r + ik E_z' \qquad (52c)$$

$$\vec{B} = \frac{c}{i\omega} \left[-ik E_\theta \, \hat{r} + (ik E_r - E_z') \, \hat{\theta} + \left(E_\theta' + \frac{1}{r} E_\theta \right) \hat{z} \right] . \qquad (52d)$$

Using Eq. (44) for the total current, Eq. (52a–c) become

$$E_z'' + \frac{2\xi + 1}{r} E_z' + \beta^2 E_z = 0 \qquad (53a)$$

$$E_\theta'' + \frac{1}{r} E_\theta' + \left[\eta^2 - \frac{1}{r^2} \right] E_\theta = 0 \qquad (53b)$$

$$\eta^2 E_r = ik \left[1 - \frac{\sigma_2^2}{k^2 c^2} \right] E_z' \qquad (53c)$$

where Eq. (53c) has already been used to eliminate E_r from Eq. (53a), and where

$$c^2 \eta^2 = \omega^2 - k^2 c^2 - \sigma_1^2$$

$$\beta^2 = k^2 \frac{-(\omega^2 - \sigma_3^2)(\omega^2 - k^2 c^2 - \sigma_1^2)}{(\omega^2 - k^2 c^2 - \sigma_1^2)(k^2 c^2 + \sigma_5^2 + \sigma_6^2) + (k^2 c^2 - \sigma_2^2)^2}$$

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and
$$\xi = \frac{1}{2} \frac{(\sigma_3^2-\sigma_4^2)(k^2c^2-\sigma_2^2) - \sigma_6^2(\omega^2-k^2c^2-\sigma_1^2)}{(\omega^2-k^2c^2-\sigma_1^2)(k^2c^2+\sigma_5^2+\sigma_6^2) + (k^2c^2-\sigma_2^2)^2} \quad .$$

For ZERO PLASMA TEMPERATURE

$$\beta^2 = k^2 \frac{(\omega^2-\sigma_3^2)(\omega^2-k^2c^2-\sigma_1^2)}{(\omega^2-k^2c^2-\sigma_1^2)(k^2c^2+\sigma_5^2) + (k^2c^2-\sigma_2^2)^2}$$

and

$$\xi = 0 \quad .$$

For ZERO BEAM TEMPERATURE AND ZERO PLASMA TEMPERATURE

$$\beta^2 = k^2 \frac{\left(\omega^2-\omega_P^2-\omega_{||}^2\frac{\omega^2}{\Omega^2}\right)\left(\omega^2-k^2c^2-\omega_\perp^2-\omega_P^2\right)}{\left(\omega^2-k^2c^2-\omega_\perp^2-\omega_P^2\right)\left(k^2c^2+\omega_\perp^2\frac{k^2V_0^2}{\Omega^2}\right) + \left(k^2c^2-\omega_\perp^2\frac{kV_0}{\Omega}\right)^2}$$

and

$$\xi = 0 \quad .$$

The solution of Eq. (53a) is straightforward. For arbitrary temperatures

$$E_z(r) = r^{-\xi} Z_\xi(\beta r) \qquad \qquad (54a)$$

where, $w = \beta r$, $Z_\xi(w)$ satisfies the Bessel equation[10]

$$Z_\xi''(w) + \frac{1}{w} Z_\xi'(w) + \left[1 - \frac{\xi^2}{w^2}\right] Z_\xi(w) = 0 \quad . \qquad (54b)$$

For the remainder of this report the zero plasma temperature limit will be investigated and so ξ will be set equal to zero. Thus the results should be valid as long as

$$\frac{|\omega + i\nu_e|^2}{k^2c^2} \gg \frac{KT_e}{mc^2} \quad \text{and} \quad \frac{|\omega + i\nu_i|^2}{k^2c^2} \gg \frac{KT_i}{Mc^2} \quad . \qquad (55)$$

Since the plasma was assumed to be non-relativistic the right-hand sides of Eq. (55) should be small. These inequalities are consistent with the neglect of the pinch field except for those modes which have $\nu_e \ll kc \gg \omega + \omega_\beta$

assuming $V_0 \simeq c$. $\xi \neq 0$ will be treated in detail in the next report of this series.

Equations (52d, 53a-c) must be solved with the proper boundary and regularity conditions. Let $\sigma * \, \delta \, (r - r_0)$ and $\vec{J} * \, \delta \, (r - r_0)$ represent the surface charge and current, respectively, on the boundary located at $r = r_0$, where $\delta \, (r - r_0)$ is the one-dimensional Dirac delta function. Using Eqs. (12d) and (12g)

$$\Delta E_r \;=\; 4\pi\sigma * \tag{56a}$$

$$\Delta B_\theta \;=\; \frac{4\pi}{c} \, J_z * \tag{56b}$$

$$\Delta B_z \;=\; - \, \frac{4\pi}{c} \, J_\theta * \tag{56c}$$

where $\quad \Delta E \;=\; E_{outside} \, - \, E_{inside}.$

All other components of the fields are continuous. From Eq. (12g)

$$- \, ik \, \Delta(B_\theta) \;=\; \frac{4\pi}{c} \, \Delta(J_r) \; - \frac{i\omega}{c} \, \Delta(E_r) \tag{56d}$$

where

$$\Delta(J_r) \;=\; \Delta(J_r^P) \; + \; \Delta(J_r^B) \;. \tag{56e}$$

Equations (56a,b,d) are combined to give

$$\Delta(J_r) \;=\; i \, (\omega \, \sigma * \, - \, k J_z *) \;. \tag{56f}$$

Let

$$\sigma * \;=\; \sigma_P^* \, + \, \sigma_B^* \tag{57a}$$

where σ_P^* is contributed by the plasma and σ_B^* is contributed by the beam. Since the radial current is only a small perturbation on the drift motion of the beam particles it is reasonable to suppose that, in the plasma frame of reference, the total surface current is due to the beam. Therefore

$$J_z^* \;=\; \sigma_B^* \, V_0$$
$$J_\theta^* \;=\; 0 \;. \tag{57b}$$

Now Eq. (56f) becomes

$$\Delta(J_r) = i\left(\omega\, \sigma_P^* + \Omega\, \sigma_B^*\right) \ . \tag{57c}$$

Assuming that σ_P^* and σ_B^* are deposited on the surface by the discontinuity of the radial components of the plasma and beam current, respectively, then using Eqs. (56e), (42), (43), and (57c)

$$4\pi\omega^2\sigma_P^* = -4\pi i\omega\Delta(J_r^P) = \Delta\left[(P_1 + P_2)\, E_r + P_3\, E_z'\right] \tag{58a}$$

$$4\pi\omega\Omega\sigma_B^* = -4\pi i\omega\Delta(J_r^B) = \Delta\left[B_1\, E_r + B_3\, E_z'\right] \ . \tag{58b}$$

Hence, from Eqs. (56a) and (58a,b)

$$\sigma_P^* + \sigma_B^* = \sigma^* = \frac{1}{4\pi}\Delta(E_r) = -\frac{i}{\omega}\Delta(J_r^P) - \frac{i}{\Omega}\Delta(J_r^B) \ . \tag{59}$$

Expressing E_r in terms of E_z' by Eq. (53c), and using the facts that $\sigma_1^2 = B_1 + P_1 + P_2$ and $\sigma_2^2 = ik\,(B_3 + P_3)$, then Eq. (59) becomes

$$\Delta(W\, E_z') = 0$$

where

$$W = \frac{\left(\omega^2 - \sigma_1^2 - \dfrac{k\,V_0}{\Omega}\,B_1\right)\left(k^2c^2 - \sigma_2^2\right) + \left(\omega^2 - k^2c^2 - \sigma_1^2\right)\left(\sigma_2^2 + ik\,\dfrac{k\,V_0}{\Omega}\,B_3\right)}{\omega^2 - k^2c^2 - \sigma_1^2}$$

where the reader is reminded that

$$B_1 = \omega_\perp^2$$

and

$$B_3 = \frac{i}{k}\,\omega_\perp^2\left[1 + \frac{\omega}{k\,U_B}\,Z_B\right] \ .$$

Therefore the regularity and boundary conditions for the $\ell = 0$ normal mode are

(1) E_r, E_θ, E_z, E_θ', E_z' must be finite at $r = 0$ and $r = \infty$,

(2) E_θ, E_θ', E_z, and $W E_z'$ are continuous at the beam–plasma interface.

(These boundary conditions are valid for arbitrary plasma temperature.)

VI DISPERSION RELATIONS FOR AN INFINITE PLASMA

The solutions of Maxwell's equations outside the beam depend upon the phases of β and η. The beam is assumed to be a cylinder of radius r_0. Henceforth the notation will be used such that $\beta = \beta_S$ for $r < r_0$, and $\beta = \beta_L$ for $r > r_0$, and similarly for η, σ_1^2, etc.

The solutions to Eqs. (53a,b) that satisfy the regularity conditions are

$$r < r_0: \qquad E_z(r) = C_1 J_0(\beta_S r) \tag{61a}$$

$$E_\theta(r) = C_4 J_1(\eta_S r) \tag{61b}$$

$$r > r_0: \quad \text{Im } \beta_L > 0, \text{ or Im } \beta_L = 0, \text{ Re } \beta_L > 0$$

$$E_z(r) = C_2 H_0^{(1)}(\beta_L r) \tag{61c}$$

$$\beta_L = 0$$

$$E_z(r) = C_3 \tag{61d}$$

$$\text{Im } \eta_L > 0, \text{ or Im } \eta_L = 0, \text{ Re } \eta_L > 0$$

$$E_\theta(r) = C_5 H_1^{(1)}(\eta_L r) \tag{61e}$$

$$\eta_L = 0$$

$$E_\theta(r) = C_6 \frac{1}{r} \tag{61f}$$

where the C_i's are constants of integration. For the opposite sign of $\text{Im } \beta_L$ the same solution and dispersion relation will be obtained except that $H_0^{(2)}$ replaces $H_0^{(1)}$. Hence only the solution for the case of $\text{Im } \beta_L > 0$ will be given explicitly, and similarly for E_θ in terms of the phase of η_L.

The continuity conditions at $r = r_0$ give the following relations where $J'(U) = \dfrac{dJ(U)}{dU}$, etc. :

Case I: $\quad$ Im $\beta_L > 0$ or Im $\beta_L = 0$, Re $\beta_L > 0$ and

$$C_1 J_0 (\beta_S r_0) = C_2 H_0^{(1)} (\beta_L r_0) \tag{62a}$$

$$C_1 W_S \beta_S J_0' (\beta_S r_0) = C_2 W_L \beta_L H_0^{(1)'} (\beta_L r_0)$$

Case II:

$$\beta_L = 0 \qquad \text{and}$$

$$C_1 J_0 (\beta_S r_0) = C_3 \tag{62b}$$

$$C_1 W_S \beta_S J_0' (\beta_S r_0) = 0$$

Case III: $\quad$ Im $\eta_L > 0$ or Im $\eta_L = 0$, Re $\eta_L > 0$ and

$$C_4 J_1 (\eta_S r_0) = C_5 H_1^{(1)} (\eta_L r_0) \tag{63a}$$

$$C_4 \eta_S J_1' (\eta_S r_0) = C_5 \eta_L H_1^{(1)'} (\eta_L r_0)$$

Case IV:

$$\eta_L = 0 \qquad \text{and}$$

$$C_4 J_1 (\eta_S r_0) = C_6 \frac{1}{r_0} \tag{63b}$$

$$C_4 \eta_S J_1' (\eta_S r_0) = - C_6 \frac{1}{r_0^2} \quad .$$

The DISPERSION RELATIONS for the case of a cold infinite plasma, as obtained from Eqs. (62, 63) and using the identity

$$z J_\nu' (z) = - \nu J_\nu (z) + z J_{\nu-1} (z) \quad ,$$

which also holds for $H_\nu^{(1,2)}$, are:

Case I: $\quad$ Im $\beta_L > 0$, or Im $\beta_L = 0$, Re $\beta_L > 0$ and

$$W_S \beta_S J_1 (\beta_S r_0) H_0^{(1)} (\beta_L r_0) = W_L \beta_L J_0 (\beta_S r_0) H_1^{(1)} (\beta_L r_0)$$

$$\tag{64a}$$

Case II: $\quad \beta_L = 0$ and

$$W_S \beta_S J_1 (\beta_S r_0) = 0 \tag{64b}$$

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Case III: $\text{Im } \eta_L > 0,$ or $\text{Im } \eta_L = 0,$ $\text{Re } \eta_L > 0$ and

$$\eta_L \, J_1 \, (\eta_S r_0) \, H_0^{(1)} (\eta_L r_0) \; = \; \eta_S \, J_0 \, (\eta_S r_0) \, H_1^{(1)} (\eta_L r_0) \tag{64c}$$

Case IV: $\eta_L = 0$ and

$$\eta_S \, J_0 \, (\eta_S r_0) \; = \; 0 \tag{64d}$$

where

$$c^2 \eta_L^2 \; = \; \omega^2 - k^2 c^2 - \omega_P^2 \; = \; c^2 \beta_L^2 \; .$$

The dispersion relations, Eqs. (64a–d), will now be discussed in a preliminary fashion. There are two main ways of analyzing these dispersion relations[2]:

(1)   specifying a real value for k and searching for values of ω satisfying the dispersion relations or

(2)   specifying real ω and searching for the corresponding values of k.

In the first case after a disturbance occurs at a particular instant, solutions are classified as to their evolvement in time: damped oscillations if $\text{Im } \omega < 0,$ oscillating solutions if $\text{Im } \omega = 0,$ and unstable solutions if $\text{Im } \omega > 0.$ In the second case, the solution represents the effect of imposing a periodic disturbance at some cross section of the beam and its evolvement along the beam for increasing z: damped, oscillating, or unstable solutions for $\text{Im } k < 0,$ $\text{Im } k = 0,$ and $\text{Im } k > 0,$ respectively. (In this case an unstable solution represents the unlimited increase, at cross sections for which $z - z_0 \rightarrow \infty,$ of the disturbance which originally took place at z_0.) It is, of course, also possible to find solutions for ω and k both having imaginary parts. The discussion here will be restricted primarily to real k and complex ω.

The dispersion relations found by BWR for the infinite beam case were classified by them as electrostatic (ES) and electromagnetic (EM):

EM: $$\omega^2 \; = \; k^2 c^2 + \omega_P^2 + \omega_\perp^2 \; = \; k^2 c^2 + \sigma_{1S}^2 \tag{65a}$$

ES: $$\omega^2 \; = \; \omega_P^2 - \omega_\parallel^2 \, \frac{\omega^2}{k^2 U_B^2} \left[1 + \frac{\Omega}{k U_B} Z_B \right] \; = \; \sigma_{3S}^2 \; . \tag{65b}$$

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These results are reproduced here by noting that a solution of Case I is $\beta_S = 0$ and $\beta_S r_0 = \alpha_{0n}$, where α_{0n} is the nth root of $J_0(\alpha) = 0$. Since $\alpha_{0n} \neq 0$, this solution is possible only for $r_0 \rightarrow \infty$ and, using the definition of β_S, Eqs. (65a, b) are obtained. (That these modes still correspond to EM and ES designation can be seen by noting that since, for $r_0 \rightarrow \infty$, E_r is given everywhere by

$$(\omega^2 - k^2 c^2 - \sigma_{1S}^2) E_r = -\frac{i}{k} \left[k^2 c^2 - \sigma_{2S}^2 \right] \beta_S \, J_1 (\beta_S r)$$

then when $\omega^2 - k^2 c^2 - \sigma_{1S}^2 = 0$ E_r can be non-zero but E_r must be zero when $\omega^2 = \sigma_{3S}^2$.) It is quite obvious that the simple ES dispersion relation of BWR, which gave the only instabilities in the infinite beam case, does not result for r_0 finite. However, it must be emphasized that it is very probable that further analysis of the complicated dispersion relations, Eqs. (64a–d), will uncover unstable solutions with growth rates comparable to the infinite beam case (cf., FGWWR). However, it should also be noted that Cases II and IV correspond to

$$c^2 \beta_L^2 = c^2 \eta_L^2 = \omega^2 - k^2 c^2 - \omega_P^2 = 0$$

which only allow damped solutions. Thus only Case I and Case III, Eqs. (64a) and (64c), need be investigated for unstable solutions.

VII DISPERSION RELATIONS FOR A PLASMA BOUNDED BY A CONDUCTING CYLINDRICAL WALL

The plasma is now assumed to extend out to a cylindrical conducting wall of radius $R_0 > r_0$. The solutions of the field equations are now

$r < r_0$:
$$E_z(r) = C_1 J_0(\beta_S r) \tag{66a}$$

$$E_\theta(r) = C_4 J_1(\eta_S r) \tag{66b}$$

$r_0 < r < R_0$: $\quad \beta_L \neq 0$
$$E_z(r) = C_2 H_0^{(1)}(\beta_L r) + C_2^! H_0^{(2)}(\beta_L r) \tag{66c}$$

$\beta_L = 0$
$$E_z(r) = C_3 + C_3^! \ln r \tag{66d}$$

$\eta_L \neq 0$
$$E_\theta(r) = C_5 H_1^{(1)}(\eta_L r) + C_5^! H_1^{(2)}(\eta_L r) \tag{66e}$$

$\eta_L = 0$
$$E_\theta(r) = C_6 \frac{1}{r} + C_6^! r \ . \tag{66f}$$

The effect of the plasma sheath at the conducting wall is neglected. (Hence under investigation here is really the physical situation where the plasma is separated from the conducting wall by a very thin layer of neutral gas.) The boundary conditions at the wall (assuming a perfect conductor) are such that the tangential component of $\vec{E}$ and the normal component of $\vec{B}$ are continuous: both are satisfied, for the $\ell = 0$ normal mode, if $E_\theta(R_0) = 0 = E_z(R_0)$.

The resulting dispersion relations are:

$\beta_L \neq 0$ and

$$\beta_L \, W_L \, J_0 \, (\beta_S \, r_0) \left[H_0^{(1)} (\beta_L R_0) \, H_1^{(2)} (\beta_L r_0) \, - \, H_0^{(2)} (\beta_L R_0) \, H_1^{(1)} (\beta_L r_0) \right]$$
$$= \beta_S \, W_S \, J_1 \, (\beta_S \, r_0) \left[H_0^{(1)} (\beta_L R_0) \, H_0^{(2)} (\beta_L r_0) \, - \, H_0^{(1)} (\beta_L r_0) \, H_0^{(2)} (\beta_L R_0) \right] \quad (67)$$

$\beta_L = 0$ and

$$W_L \, J_0 \, (\beta_S \, r_0) \; = \beta_S \, W_S \, J_1 \, (\beta_S \, r_0) \, r_0 \ln \frac{R_0}{r_0} \tag{68}$$

$\eta_L \neq 0$ and

$$\eta_L \, J_1 \, (\eta_S \, r_0) \left[H_0^{(1)} (\eta_L r_0) \, H_1^{(2)} (\eta_L R_0) \, - \, H_0^{(2)} (\eta_L r_0) \, H_1^{(1)} (\eta_L R_0) \right]$$
$$= \eta_S \, J_0 \, (\eta_S \, r_0) \left[H_1^{(1)} (\eta_L r_0) \, H_1^{(2)} (\eta_L R_0) \, - \, H_1^{(2)} (\eta_L r_0) \, H_1^{(1)} (\eta_L R_0) \right] \quad (69)$$

and finally

$\eta_L = 0$ and

$$\eta_S \, (R_0^2 \, - \, r_0^2) \, J_0 \, (\eta_S \, r_0) \; = \; - \, 2 \, r_0 \, J_1 \, (\eta_S \, r_0) \; . \tag{70}$$

It is important to note that Eqs. (68) and (70) have only damped solutions since $\beta_L = 0$ and $\eta_L = 0$ both imply $\omega^2 = k^2 c^2 + \omega_P^2$. It is also interesting to note that the EM mode is still present as a solution of Eq. (69).

VIII DISPERSION RELATIONS FOR A FINITE PLASMA BOUNDED BY VACUUM

In vacuum, Maxwell's equations for the $1 = 0$ mode are

$$E_z'' + \frac{1}{r} E_z' + \eta_\nu^{\,2} E_z = 0 \tag{71a}$$

$$E_\theta'' + \frac{1}{r} E_\theta' + \left[\eta_\nu^{\,2} - \frac{1}{r^2} \right] E_\theta = 0 \tag{71b}$$

$$\eta_\nu^2 E_r = i k E_z' \tag{71c}$$

where $\quad c^2 \eta_\nu^2 = \omega^2 - k^2 c^2$.

The plasma is assumed to extend to R_0 beyond which vacuum extends to infinity. The solutions for the regions for which $r < R_0$ are given by Eqs. (66a–f). In the vacuum the solutions are:

$$r > R_0 : \qquad \mathrm{Im}\,\eta_\nu > 0 \ , \ \text{ or } \ \mathrm{Im}\,\eta_\nu = 0 \ , \ \mathrm{Re}\,\eta_\nu > 0 \text{ and}$$

$$E_z(r) = C_7 H_0^{(1)}(\eta_\nu r) \tag{72a}$$

$$E_\theta(r) = C_8 H_1^{(1)}(\eta_\nu r) \tag{72b}$$

$$\text{and} \qquad \eta_\nu = 0 \ , \qquad E_z(r) = C_{11} \tag{72c}$$

$$E_\theta(r) = C_{12} \frac{1}{r} \ . \tag{72d}$$

Since it has been assumed that in its own rest frame the plasma produces no surfaces current then, at the plasma-vacuum interface,

$$\frac{\omega^2 - \sigma_1^2 - \sigma_2^2}{\omega^2 - k^2 c^2 - \sigma_1^2} \ E_z' \ ,$$

instead of WE_z' , is continuous.

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The resulting dispersion relations are

$\beta_L \neq 0$ and $\text{Im}\,\eta_\nu > 0$ or $\text{Im}\,\eta_\nu = 0$, $\text{Re}\,\eta_\nu > 0$:

$$\beta_L \, W_L \, J_0 \, (\beta_S r_0) \left[\beta_L \, \frac{\omega^2 - \sigma_{1L}^2 - \sigma_{2L}^2}{\omega^2 - k^2 c^2 - \sigma_{1L}^2} \, H_0^{(1)} (\eta_\nu R_0) \left\{ H_1^{(1)} (\beta_L r_0) \, H_1^{(2)} (\beta_L R_0) \right. \right.$$

$$\left. - H_1^{(1)} (\beta_L R_0) \, H_1^{(2)} (\beta_L r_0) \right\} + \frac{\omega^2}{\omega^2 - k^2 c^2} \, \eta_\nu \, H_1^{(1)} (\eta_\nu R_0)$$

$$\left. \left\{ H_0^{(1)} (\beta_L R_0) \, H_1^{(2)} (\beta_L r_0) \, - \, H_0^{(2)} (\beta_L R_0) \, H_1^{(1)} (\beta_L r_0) \right\} \right]$$

$$\tag{73}$$

$$= \beta_S \, W_S \, J_1 \, (\beta_S r_0) \left[\beta_L \, \frac{\omega^2 - \sigma_{1L}^2 - \sigma_{2L}^2}{\omega^2 - k^2 c^2 - \sigma_{1L}^2} \, H_0^{(1)} (\eta_\nu R_0) \left\{ H_0^{(1)} (\beta_L r_0) \, H_1^{(2)} (\beta_L R_0) \right. \right.$$

$$\left. - H_0^{(2)} (\beta_L r_0) \, H_1^{(1)} (\beta_L R_0) \right\} + \frac{\omega^2}{\omega^2 - k^2 c^2} \, \eta_\nu \, H_1^{(1)} (\eta_\nu R_0)$$

$$\left. \left\{ H_0^{(1)} (\beta_L R_0) \, H_0^{(2)} (\beta_L r_0) \, - \, H_0^{(1)} (\beta_L r_0) \, H_0^{(2)} (\beta_L R_0) \right\} \right]$$

$\beta_L = 0$ and $\text{Im}\,\eta_\nu > 0$ or $\text{Im}\,\eta_\nu = 0$, $\text{Re}\,\eta_\nu > 0$:

$$W_L \, J_0 \, (\beta_S r_0) \, \frac{\omega^2}{\omega^2 - k^2 c^2} \, \eta_\nu H_1^{(1)} (\eta_\nu R_0) = \beta_S \, W_S \, J_1 \, (\beta_S r_0) \left[\frac{\omega^2 - \sigma_{1L}^2 - \sigma_{2L}^2}{\omega^2 - k^2 c^2 - \sigma_{1L}^2} \, \frac{r_0}{R_0} \, H_0^{(1)} (\eta_\nu R_0) \right.$$

$$\tag{74}$$

$$\left. + \frac{\omega^2}{\omega^2 - k^2 c^2} \eta_\nu \, r_0 \ln \frac{R_0}{r_0} \, H_1^{(1)} (\eta_\nu R_0) \right]$$

$\beta_L \neq 0$ and $\eta_\nu = 0$:

$$\beta_L \, W_L \, J_0 \, (\beta_S r_0) \left[H_1^{(1)} (\beta_L R_0) \, H_1^{(2)} (\beta_L r_0) \, - \, H_1^{(1)} (\beta_L r_0) \, H_1^{(2)} (\beta_L R_0) \right]$$

$$\tag{75}$$

$$= \beta_S \, W_S \, J_1 \, (\beta_S r_0) \left[H_1^{(1)} (\beta_L R_0) \, H_0^{(2)} (\beta_L r_0) \, - \, H_0^{(1)} (\beta_L r_0) \, H_1^{(2)} (\beta_L R_0) \right] \, .$$

$\eta_L \neq 0$ and $\operatorname{Im} \eta_\nu > 0$ or $\operatorname{Im} \eta_\nu = 0$, $\operatorname{Re} \eta_\nu > 0$:

$$\eta_L J_1(\eta_S r_0) \left[\eta_\nu H_0^{(1)}(\eta_\nu R_0) \left\{ H_0^{(1)}(\eta_L r_0) H_1^{(2)}(\eta_L R_0) - H_0^{(2)}(\eta_L r_0) H_1^{(1)}(\eta_L R_0) \right\} \right.$$

$$\left. + \eta_L H_1^{(1)}(\eta_\nu R_0) \left\{ H_0^{(2)}(\eta_L r_0) H_0^{(1)}(\eta_L R_0) - H_0^{(1)}(\eta_L r_0) H_0^{(2)}(\eta_L R_0) \right\} \right]$$

$$\tag{76}$$

$$= \eta_S J_0(\eta_S r_0) \left[\eta_\nu H_0^{(1)}(\eta_\nu R_0) \left\{ H_1^{(1)}(\eta_L r_0) H_1^{(2)}(\eta_L R_0) - H_1^{(1)}(\eta_L R_0) H_1^{(2)}(\eta_L r_0) \right\} \right.$$

$$\left. + \eta_L H_1^{(1)}(\eta_\nu R_0) \left\{ H_0^{(1)}(\eta_L R_0) H_1^{(2)}(\eta_L r_0) - H_0^{(2)}(\eta_L R_0) H_1^{(1)}(\eta_L r_0) \right\} \right]$$

$\eta_L = 0$ and $\operatorname{Im} \eta_\nu > 0$ or $\operatorname{Im} \eta_\nu = 0$, $\operatorname{Re} \eta_\nu > 0$:

$$J_1(\eta_S r_0) \, \eta_\nu \, r_0 \, 2 \, H_0^{(1)}(\eta_\nu R_0) = \eta_S J_0(\eta_S r_0) \left[2 R_0 H_1^{(1)}(\eta_\nu R_0) \right.$$

$$\left. - \eta_\nu (R_0^2 - r_0^2) H_0^{(1)}(\eta_\nu R_0) \right]$$

$$\tag{77}$$

$\eta_L \neq 0$ and $\eta_\nu = 0$:

$$\eta_L J_1(\eta_S r_0) \left[H_0^{(1)}(\eta_L R_0) H_0^{(2)}(\eta_L r_0) - H_0^{(1)}(\eta_L r_0) H_0^{(2)}(\eta_L R_0) \right]$$

$$\tag{78}$$

$$= \eta_S J_0(\eta_S r_0) \left[H_0^{(1)}(\eta_L R_0) H_1^{(2)}(\eta_L r_0) - H_0^{(2)}(\eta_L R_0) H_1^{(1)}(\eta_L r_0) \right]$$

The cases of $\eta_\nu = 0$ and $\beta_L = 0$ or $\eta_L = 0$ are not possible for zero plasma temperature.

It is important to notice that Eqs. (74) and (77) only admit damped solutions since $\beta_L = 0 = \eta_L$. Also Eqs. (75) and (78) admit only oscillatory solutions since $\eta_\nu = 0$ implies $\omega^2 = k^2 c^2$. It is also interesting to note that the EM mode is still present as a solution to Eq. (76).

IX CONCLUDING REMARKS

The main results of this report are those dispersion relations that may possibly have unstable solutions: Equations (64a) and (64c) for an infinite plasma, Eqs. (67) and (69) for a plasma enclosed by a cylindrical conducting wall, and Eqs. (73) and (76) for a plasma bounded by an infinite vacuum.

In a subsequent report, these relations will be analyzed in order to ascertain whether there actually are any unstable solutions. Especially interesting will be the answer to the following question: Are any unstable modes for an infinite plasma stablized by a bounding wall or vacuum or, possibly, vice-versa? The dispersion relations will also be generalized to include some of the effects omitted in this report: plasma temperature, the pinch field, beam collisions and higher normal modes.

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APPENDIX

VECTOR IDENTITIES IN CYLINDRICAL COORDINATES

Listed in this appendix are the vector identities necessary in order to obtain the beam and plasma current responses, Eqs. (42) and (43). These identities are given in cylindrical coordinates using the notation $E' = \dfrac{dE}{dr}$, $E'' = \dfrac{d^2 E}{dr^2}$, etc. and are for $\vec{E}$ being θ independent (i.e., the $\ell = 0$ normal mode) only.

Let

$$\vec{F}(r,\vec{v}) = \vec{E}(r) + \frac{\vec{V_0} + \vec{v}}{-i\omega} \times \left[(\nabla_\perp + ik\,\hat{z}) \times \vec{E}(r) \right] \tag{A-1}$$

where the factor, $e^{i(kz - \omega t)}$, is understood on both sides of (A-1) and in all the formulas to follow. For the plasma $V_0 = 0$.

Thus

$$\vec{F}(r,\vec{v}) = \hat{r} \left[\frac{\Omega - kv_z}{\omega} E_r + \frac{V_0 + v_z}{i\omega} E_z' + \frac{v_\theta}{i\omega} \left(E_\theta' + \frac{1}{r} E_\theta \right) \right]$$

$$+ \hat{\theta} \left[\frac{\Omega - kv_z}{\omega} E_\theta - \frac{v_r}{i\omega} \left(E_\theta' + \frac{1}{r} E_\theta \right) \right] \tag{A-2}$$

$$+ \hat{z} \left[E_z + \frac{v_r}{i\omega} (ik E_r - E_z') + \frac{v_\theta}{i\omega} ik E_\theta \right]$$

$$(\vec{v}_\perp \cdot \nabla_\perp)\vec{F} = \hat{r} \left[v_r E_r' - v_\theta \frac{1}{r} E_\theta + \frac{v_r v_\theta}{i\omega} \left(E_\theta'' + \frac{2}{r} E_\theta' \right) \right.$$

$$+ \frac{v_r (V_0 + v_z)}{i\omega} (E_z'' - ik E_r') + \frac{v_\theta (V_0 + v_z)}{i\omega} \frac{ik}{r} E_\theta \right]$$

$$+ \hat{\theta} \left[v_r E_\theta' + v_\theta \frac{1}{r} E_r - \frac{v_r^2}{i\omega} \left(E_\theta'' + \frac{1}{r} E_\theta' - \frac{1}{r^2} E_\theta \right) + \frac{v_r (V_0 + v_z)}{i\omega} (-ik E_\theta') \right.$$

$$+ \frac{v_\theta (V_0 + v_z)}{i\omega} \frac{1}{r} (E_z' - ik E_r) + \frac{v_\theta^2}{i\omega} \frac{1}{r} \left(E_\theta' + \frac{1}{r} E_\theta \right) \right] \tag{A-3}$$

$$+ \hat{z} \left[v_r E_z' + \frac{v_r^2}{i\omega} (ik E_r' - E_z'') + \frac{v_r v_\theta}{i\omega} ik E_\theta' \right]$$

The following is only needed for the plasma so V_0 has been set equal to zero:

$$(\vec{v}_\perp \cdot \nabla_\perp)^2 \vec{F} = \hat{r}\left[v_r^2 E_r'' - v_\theta^2 \frac{1}{r^2} E_r - v_r v_\theta \frac{1}{r}\left(2 E_\theta' - \frac{1}{r} E_\theta \right) + v_r^2 v_z (E_z''' - ik E_r'' \right.$$

$$+ v_r^2 v_\theta \left(E_\theta''' + \frac{3}{r} E_\theta'' - \frac{1}{r^2} E_\theta' - \frac{1}{r^3} E_\theta \right) + v_r v_\theta v_z \frac{ik}{r}\left(2 E_\theta' - \frac{1}{r} E_\theta \right)$$

$$\left. + v_z v_\theta^2 \frac{1}{r^2} (ik E_r - E_z') - v_\theta^3 \frac{1}{r^2}\left(E_\theta' + \frac{1}{r} E_\theta \right) \right]$$

$$+ \hat{\theta}\left[v_r^2 E_\theta'' - v_\theta^2 \frac{1}{r^2} E_\theta + v_r v_\theta \frac{1}{r}\left(2 E_r' - \frac{1}{r} E_r \right) \right. \qquad \text{(A-4)}$$

$$- v_r^3 \left(E_\theta''' + \frac{1}{r} E_\theta'' - \frac{2}{r^2} E_\theta' + \frac{2}{r^3} E_\theta \right) - v_r^2 v_z \, ik \, E_\theta''$$

$$+ v_r v_\theta v_z \left(E_z''' + \frac{1}{r} E_z'' - ik\left[E_r'' + \frac{1}{r} E_r' \right] \right)$$

$$\left. + v_r v_\theta^2 \frac{2}{r}\left(E_\theta'' + \frac{1}{r} E_\theta' - \frac{1}{r^2} E_\theta \right) + v_\theta^2 v_z \frac{ik}{r^2} E_\theta \right]$$

$$+ \hat{z}\left[v_r^2 E_z'' + v_r^3 (ik E_r'' - E_z''') + v_r^2 v_\theta \, ik \, E_\theta'' \right] \quad .$$

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